2023-24 MATH2048: Honours Linear Algebra II

Homework 9

Due: 2023-11-27 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. Let V be a finite-dimensional inner product space, and let T be a linear operator on V. If T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

2. Let V be an inner product space, and let T be a linear operator on V. Prove the following results.

   (a) $R(T^*)^{\perp} = N(T)$.

   (b) If V is finite-dimensional, then $R(T^*) = N(T)^{\perp}$

3. Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. If W is T-invariant, then W is also T^*-invariant.

4. Let T be a normal operator on a finite-dimensional inner product space V. Then $N(T) = N(T^*)$ and $R(T) = R(T^*)$.

5. Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Prove that

   (a) $U(W) = W$

   (b) $W^{\perp}$ is U-invariant.

The following are extra recommended exercises not included in homework.

1. Suppose that A is an $m \times n$ matrix in which no two columns are identical. Then, A^*A is a diagonal matrix if and only if every pair of columns of A is orthogonal.

2. Let $V = W \oplus W^{\perp}$ be an inner product space and T be the projection on W along $W^{\perp}$. Then $T = T^*$.

3. Let T be a linear operator on a finite-dimensional vector space V. Prove the following results.

   (a) $N(T^*T) = N(T)$. Deduce that $\text{rank}(T^*T) = \text{rank}(T)$.

   (b) $\text{rank}(T) = \text{rank}(T^*)$. Deduce from (a) that $\text{rank}(TT^*) = \text{rank}(T)$.

   (c) For any $n \times n$ matrix A, $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$.

4. Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following results.

   (a) If T is self-adjoint, then $T|_W$ is self-adjoint.

   (b) $W^\perp$ is T^*-invariant.

   (c) If W is both T- and T^*-invariant, then $(T|_W)^* = (T^*)|_W$.

   (d) If W is both T- and T^*-invariant and T is normal, then $T|_W$ is normal.

5. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that for all $x \in V$,

$$\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2$$

   Deduce that $T - iI$ is invertible and that $[(T - iI)^{-1}]^* = (T + iI)^{-1}$.

6. If T is a unitary operator on a finite-dimensional inner product space V, then T has a unitary square root; that is, there exists a unitary operator U such that $T = U^2$.